Solution 3 by Arkady Alt, San Jose, CA

First note that $f(x) \cdot f(y) \neq -1$ for any $x, y \in R$.

Since $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = \dfrac{2f\left(\frac{x}{2}\right)}{1 + f^2\left(\frac{x}{2}\right)} \Rightarrow |f(x)| = \dfrac{2\left|f\left(\frac{x}{2}\right)\right|}{1 + \left|f\left(\frac{x}{2}\right)\right|^2}$

and then we have $\left(\left|f\left(\frac{x}{2}\right)\right| - 1\right)^2 \geq 0 \iff \dfrac{2\left|f\left(\frac{x}{2}\right)\right|}{1 + \left|f\left(\frac{x}{2}\right)\right|^2} \leq 1 \iff |f(x)| \leq 1.$

If we suppose $|f(x_0)| = 1,$ for some $x_0,$ then $\left|f\left(\frac{x_0}{2}\right)\right| = 1$ and $f(x)$ becomes a constant

function. Indeed, if $f(x_0) = 1,$ then for any $x \in R$ we have $f(x + x_0) = \dfrac{f(x) + 1}{1 + f(x)} = 1,$

because $f(x) = f(x) \cdot f(x_0) \neq -1.$

If $f(x_0) = -1,$ then for any $x \in R$ we have $f(x + x_0) = \dfrac{f(x) - 1}{1 - f(x)} = -1,$

because $-f(x) = f(x) \cdot f(x_0) \neq -1.$ Thus, $|f(x)| < 1 \iff -1 < f(x) < 1$ for any x.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Statesboro, GA; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Buzău, Romania, and the proposer.

5205: *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the sum,

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln \frac{n+1}{n}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For each integer $m > 1$, is easy to prove by induction that

$$\sum_{n=1}^{m} \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n}\right) \ln \frac{n+1}{n}$$

$$= \left(1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}\right)\ln(m+1)+\sum_{n=2}^{m}\frac{(-1)^n\ln n}{n}.$$

Since

$$\left|1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}-\ln 2\right|$$

$$= \frac{1}{m+1}\left(1-\frac{m+1}{m+2}+\frac{m+1}{m+3}-\frac{m+1}{m+4}+\cdots\right)<\frac{1}{m+1},$$

so

$$\lim_{m\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}-\ln 2\right)\ln(m+1)=0.$$

It is known [E. R. Hansen: *A Table of Series and Products,* Prentice-Hall, Inc., 1975, p. 288 entry (44.1.8)] that $\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{n}=\gamma\ln 2-\frac{(\ln 2)^2}{2}$, where γ is Euler's constant. Hence the sum of the problem equals $\gamma\ln 2-\frac{(\ln 2)^2}{2}=0.1598\ldots.$

Solution 2 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy

By writing $q_n=1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{n}-\ln 2$ the series is

$$\sum_{n=1}^{\infty}q_n\ln\frac{n+1}{n}=\sum_{n=1}^{\infty}\left((q_n\ln(n+1)-q_{n-1}\ln n)+\ln n(q_{n-1}-q_n)\right)$$

$$\sum_{n=1}^{\infty}(q_n\ln(n+1)-q_{n-1}\ln n)=\lim_{n\to\infty}q_n\ln(n+1).$$

The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

is Leibniz and converges to $\ln 2$ thus it satisfies

$$\left|\ln 2-\sum_{n=1}^{r}\frac{(-1)^{n-1}}{n}\right|\leq\frac{1}{r+1}.$$

Since this is a well known property of all Leibniz series present in all books on the subject, we omit it. The immediate consequence is

$$\lim_{n\to\infty}q_n\ln(n+1)=0.$$

We remain with

$$\sum_{n=1}^{\infty} \ln n(q_{n-1} - q_n) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln n = \gamma \ln 2 - \frac{1}{2} \ln^2 2$$

where γ is the Euler–Mascheroni constant. Also $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln n = \gamma \ln 2 - \frac{1}{2} \ln^2 2$ is a well known result. Nevertheless we write it here. For $p \geq 4$,

$$\sum_{k=2}^{2p} (-1)^k \frac{\ln k}{k} = \sum_{k=1}^{p} \frac{\ln 2}{2k} + \sum_{k=1}^{p} \frac{\ln k}{2k} - \sum_{k=1}^{p-1} \frac{\ln(2k+1)}{2k+1}.$$

$$-\sum_{k=1}^{p-1} \frac{\ln(2k+1)}{2k+1} = -\sum_{k=2}^{2p-1} \frac{\ln k}{k} + \sum_{k=1}^{p-1} \frac{\ln(2k)}{2k} = -\sum_{k=2}^{2p-1} \frac{\ln k}{k} + \sum_{k=1}^{p-1} \frac{\ln 2}{2k} + \sum_{k=1}^{p-1} \frac{\ln k}{2k}.$$

By summing we get

$$\sum_{k=1}^{p-1} \frac{\ln 2}{k} + \frac{\ln 2}{2p} + \frac{\ln p}{2p} - \sum_{k=p}^{2p-1} \frac{\ln k}{k}.$$

Now we employ the well known

$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$. Moreover we observe that

$$\int_{p}^{2p} \frac{\ln x}{x} dx \leq \sum_{k=p}^{2p-1} \frac{\ln k}{k} = \int_{p-1}^{2p-1} \frac{\ln x}{x} dx,$$

(*Editor's note:* We note that the function $\frac{\ln x}{x}$ is decreasing for $x \geq e$. So $\int_{k}^{k+1} \frac{\ln x}{x} dx \leq \frac{\ln k}{k} \leq \int_{k-1}^{k} \frac{\ln x}{x} dx$. The claimed inequalities follow by summing over k from $k = p$ to $k = 2p - 1$.)

thus

$$\begin{aligned}
\sum_{k=p}^{2p-1} \frac{\ln k}{k} &= \int_{p}^{2p-1} \frac{\ln x}{x} dx + o(1) = \frac{1}{2}(\ln^2(2p-1) - \ln^2 p) + o(1) \\
= \quad & \frac{\ln^2 2}{2} + \frac{\ln^2 p}{2} + \ln 2 \ln p + \ln(2p) \ln(1 - \frac{1}{2p}) + \frac{1}{2} \ln^2(1 - \frac{1}{2p}) - \frac{\ln^2 p}{2} + o(1) \\
= \quad & \frac{\ln^2 2}{2} + \ln 2 \ln p + o(1).
\end{aligned}$$

We get

$$\sum_{k=1}^{p-1}\frac{\ln 2}{k}+\frac{\ln 2}{2p}+\frac{\ln p}{2p}-\sum_{k=p}^{2p-1}\frac{\ln k}{k}=\ln 2(\ln(p-1)+\gamma)-\frac{\ln^2 2}{2}-\ln 2\ln p+o(1)$$

$$=\quad \gamma\ln 2-\frac{\ln^2 2}{2},\ \text{as } p\to\infty.$$

Solution 3 by Anastasios Kotronis, Athens, Greece

We set

$$f_m(x)=\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\qquad x<1,$$

and we wish to find

$$\lim_{m\to+\infty} f_m(-1).$$

For $x<1$ we have

$$\begin{aligned}
f_m'(x) &= \left(\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\right)'\\
&= \sum_{n=1}^{m}\left(-\sum_{k=0}^{n-1}x^k+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&= \sum_{n=1}^{m}\left(-\frac{1-x^n}{1-x}+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&= \frac{1}{1-x}\sum_{n=1}^{m}x^n\left(\ln(n+1)-\ln n\right)\\
&= \frac{1}{1-x}\left(\sum_{n=2}^{m}\left(x^{n-1}-x^n\right)\ln n+x^m\ln(m+1)\right)\\
&= \sum_{n=2}^{m}x^{n-1}\ln n+\frac{x^m}{1-x}\ln(m+1).
\end{aligned}$$

So we integrate from 0 to y, where $y<1$, to get

$$f_m(y)=\sum_{n=2}^{m}\frac{y^n}{n}\ln n+\ln(m+1)\int_0^y\frac{x^m}{1-x}\,dx$$

and set $y=-1$ to get

$$
\begin{aligned}
f_m(-1) &= \sum_{n=2}^{m} \frac{(-1)^n}{n} \ln n + \ln(m+1) \int_0^{-1} \frac{x^m}{1-x}\, dx \\
&\overset{x=-t}{=\!=\!=} \sum_{n=2}^{m} \frac{(-1)^n}{n} \ln n + (-1)^{m+1} \ln(m+1) \int_0^{1} \frac{t^m}{1+t} dt \\
&= A_m + (-1)^{m+1} \ln(m+1) B_m. \qquad (1)
\end{aligned}
$$

Now integrating by parts,

$$
\begin{aligned}
B_m &= \left. \frac{t^{m+1}}{(m+1)(1+t)} \right|_0^1 + \frac{1}{m+1} \int_0^1 \frac{t^{m+1}}{(1+t)^2}\, dt \\
&\le \frac{1}{2(m+1)} + \frac{1}{m+1} \int_0^1 \frac{1}{(1+t)^2}\, dt \\
&= \frac{1}{m+1} < \frac{1}{m} \qquad (2)
\end{aligned}
$$

and for A_m, since it converges from Leibniz Criterion, (see: http://mathworld.wolfram.com/Leibniz Criterion.html) we can write

$$
\lim_{m\to+\infty} A_m = \lim_{m\to+\infty} A_{2m}
$$

and

$$
\begin{aligned}
A_{2m} &= \sum_{n=1}^{2m} \frac{(-1)^n}{n} \ln n \\
&= \sum_{n=1}^{m} \frac{\ln 2n}{2n} - \sum_{n=1}^{m} \frac{\ln(2n-1)}{2n-1} \\
&= \frac{\ln 2}{2} \sum_{n=1}^{m} \frac{1}{n} + \frac{1}{2} \sum_{n=1}^{m} \frac{\ln n}{n} - \left(\sum_{n=1}^{2m} \frac{\ln n}{n} - \sum_{n=1}^{m} \frac{\ln 2n}{2n} \right) \\
&= \ln 2 H_m + \sum_{n=1}^{m} \frac{\ln n}{n} - \sum_{n=1}^{2m} \frac{\ln n}{n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln(m+n)}{m+n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln m + \ln(1+n/m)}{m+n} \\
&= \ln 2 H_m - \ln m (H_{2m} - H_m) - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m}
\end{aligned}
$$

$$= \quad H_m \ln(2m) - H_{2m}\ln m - \frac{1}{m}\sum_{n=1}^{m}\frac{\ln(1+n/m)}{1+n/m}$$

$$\overset{H_m=\ln m+\gamma+O(1/m)}{=} \quad \gamma\ln 2 + O(1/m) - \frac{1}{m}\sum_{n=1}^{m}\frac{\ln(1+n/m)}{1+n/m} \tag{3}$$

Now with (2) and (3), (1) will give

$$f_m(-1) \to \gamma\ln 2 - \int_0^1 \frac{\ln(1+x)}{1+x}\,dx = \gamma\ln 2 - \frac{\ln^2 2}{2}.$$

Comment: In fact, one can easily show that

$$\frac{1}{m}\sum_{n=1}^{m}\frac{\ln(1+n/m)}{1+n/m} = \frac{\ln^2 2}{2} + O(1/m), \quad \text{so}$$

$$\sum_{n=1}^{m}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^{n-1}}{n}-\ln 2\right)\cdot\ln\left(\frac{n+1}{n}\right) = \gamma\ln 2 - \frac{\ln^2 2}{2} + O\left(m^{-1}\ln m\right).$$

Editor's comment: The sum in (3) is a Riemann sum whose limit as m tends to infinity equals the Riemann integral.

Solution 4 by Arkady Alt, San Jose, CA

Let $h_n = \sum\limits_{k=1}^{n}\frac{1}{k}$, $a_n = \sum\limits_{k=1}^{n}\frac{(-1)^{k-1}}{k} - \ln 2$, and $S = \sum\limits_{n=1}^{\infty} a_n \ln\frac{n+1}{n}$.

Note that

$$\begin{aligned}
\sum_{k=1}^{n} a_k \ln\frac{k+1}{k} &= \sum_{k=1}^{n} a_k\left(\ln(k+1)-\ln k\right)\\
&= \sum_{k=1}^{n} a_k\ln(k+1) - \sum_{k=1}^{n} a_k\ln k\\
&= \sum_{k=2}^{n+1} a_{k-1}\ln k - \sum_{k=2}^{n} a_k\ln k\\
&= a_n\ln(n+1) - \sum_{k=2}^{n}(a_k-a_{k-1})\ln k\\
&= a_n\ln(n+1) - \sum_{k=2}^{n}\frac{(-1)^{k-1}\ln k}{k}\\
&= a_n\ln(n+1) + \sum_{k=2}^{n}\frac{(-1)^{k}\ln k}{k}.
\end{aligned}$$

First we will prove $\lim_{n\to\infty} a_n \ln(n+1) = 0$.

Since $a_{2n+1} = a_{2n} + \dfrac{1}{2n+1}$ then it suffices to prove

$$\lim_{n\to\infty} a_{2n} \ln(2n+1) = 0.$$

We have $a_{2n} = h_{2n} - h_n - \ln 2$ and, since $\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where $\gamma = \lim_{n\to\infty} (h_n - \ln n)$ is Euler's constant, then

$$\begin{aligned}
& \ln 2n - \ln(n+1) - \ln 2 < a_{2n} < \ln(2n+2) - \ln n - \ln 2 \\
\iff \quad & -\ln \frac{n+1}{n} < a_{2n} < \ln \frac{n+1}{n} \\
\iff \quad & |a_{2n}| < \ln \frac{n+1}{n} < \frac{1}{n} \\
& \left(1 + \frac{1}{n}\right)^n < e \iff \ln \frac{n+1}{n} < \frac{1}{n}.
\end{aligned}$$

Hence, $0 < |a_{2n}| \ln(2n+1) < \frac{\ln(2n+1)}{n}$ yields $\lim_{n\to\infty} \frac{\ln(2n+1)}{n} = 0$, and, therefore $\lim_{n\to\infty} a_{2n} \ln(2n+1) = 0$.

Thus, $S = \lim_{n\to\infty} \sum_{k=2}^{n} s_n$, where $s_n := \sum_{k=2}^{n} \frac{(-1)^k \ln k}{k}$.

Since $s_{2n+1} = s_{2n} - \frac{\ln(2n+1)}{2n+1}$ and $\lim_{n\to\infty} \frac{\ln(2n+1)}{2n+1} = 0$ then $S = \lim_{n\to\infty} s_{2n}$.

Let $b_n := \sum_{k=1}^{n} \frac{\ln k}{k}$ then

$$\begin{aligned}
s_{2n} &= \sum_{k=1}^{2n} \frac{(-1)^k \ln k}{k} \\
&= \sum_{k=1}^{n} \frac{\ln 2k}{2k} - \sum_{k=1}^{n} \frac{\ln(2k-1)}{2k-1} \\
&= 2\sum_{k=1}^{n} \frac{\ln 2k}{2k} - \sum_{k=1}^{2n} \frac{\ln k}{k} \\
&= \sum_{k=1}^{n} \frac{\ln 2k}{k} - b_{2n} \\
&= \sum_{k=1}^{n} \frac{\ln 2}{k} + \sum_{k=1}^{n} \frac{\ln k}{k} - b_{2n} \\
&= \ln 2 \cdot h_n + b_n - b_{2n}.
\end{aligned}$$

Consider now two sequences $\left(b_n - \frac{\ln^2(n+1)}{2}\right)_{n\geq 1}$ and $\left(b_n - \frac{\ln^2 n}{2}\right)_{n\geq 1}$.

Since $b_n - \frac{\ln^2(n+1)}{2}$ is increasing and $b_n - \frac{\ln^2 n}{2}$ is decreasing in n then

$$b_1 - \frac{\ln^2 2}{2} \leq b_n - \frac{\ln^2(n+1)}{2} < b_n - \frac{\ln^2 n}{2} \leq b_1$$

and, therefore, both sequences converges to the same limit.

Let $\delta = \lim_{n\to\infty}\left(b_n - \frac{\ln^2(n+1)}{2}\right) = \lim_{n\to\infty}\left(b_n - \frac{\ln^2 n}{2}\right)$ then

$$b_n - \frac{\ln^2(n+1)}{2} < \delta < b_n - \frac{\ln^2 n}{2} \iff \frac{\ln^2 n}{2} + \delta < b_n < \frac{\ln^2(n+1)}{2} + \delta, n \in N.$$

Hence,

$$\begin{aligned} \frac{\ln^2 2n - \ln^2(n+1)}{2} &< b_{2n} - b_n < \frac{\ln^2(2n+2) - \ln^2 n}{2} \iff \\ \beta_n &< b_{2n} - b_n - \ln 2 \cdot \ln n < \alpha_n, \end{aligned}$$

where $\alpha_n = \dfrac{\ln^2(2n+2) - \ln^2 n}{2} - \ln 2 \cdot \ln n$ and $\beta_n = \dfrac{\ln^2 2n - \ln^2(n+1)}{2} - \ln 2 \cdot \ln n$.

Noting that

$\dfrac{\ln^2 2n - \ln^2 n}{2} - \ln 2 \cdot \ln n = \dfrac{\ln 2(\ln 2 + 2\ln n)}{2} - \ln 2 \cdot \ln n = \dfrac{\ln^2 2}{2}$, we obtain

$$\lim_{n\to\infty}\left(\alpha_n - \frac{\ln^2 2}{2}\right) = \lim_{n\to\infty}\left(\frac{\ln^2(2n+2) - \ln^2 2n}{2}\right) = \frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)\ln(4n(n+1)) = 0, \text{ and}$$

$$\lim_{n\to\infty}\left(\beta_n - \frac{\ln^2 2}{2}\right) = \lim_{n\to\infty}\frac{\ln^2 n - \ln^2(n+1)}{2} = -\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)\ln(n(n+1)) = 0.$$

This gives us

$$\lim_{n\to\infty}(b_{2n} - b_n - \ln 2 \cdot \ln n) = \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \frac{\ln^2 2}{2}.$$

Since $\lim_{n\to\infty}(h_n - \ln n) = \gamma$ then

$$\begin{aligned} S &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot h_n) \\ &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot \ln n + \ln 2 \cdot (h_n - \ln n)) \\ &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot \ln n) + \lim_{n\to\infty}\ln 2 \cdot (h_n - \ln n) \end{aligned}$$

$$= \ln 2 \cdot \left(\gamma - \frac{\ln 2}{2}\right).$$

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Editor's comment: Mea Culpa once again. I inadvertently gave credit to David Stone and John Hawkins for having solved problem 5199 when they should have been credited for having solved 5198. And I inadvertently forgot to acknowledge **Achilleas Sinefakopoulos of Larissa, Greece** for having correctly solved 5184.